Growth kinetics in systems with local symmetry

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The phase-transition kinetics of Ising gauge models are investigated. Despite the absence of a local order parameter, relevant topological excitations that control the ordering kinetics can be identified. Dynamical scaling holds in the approach to equilibrium, and the growth of a typical length scale is characteristic of a universality class with $L(t)\sim(t/\ln t)^{1/2}$. We suggest that the asymptotic kinetics of the two-dimensional (2D) Ising gauge model is dual to that of the 2D annihilating random walks, a process also known as the diffusion reaction $A+A\rightarrow$(inert).

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In the approach to equilibrium, systems undergoing phase transitions typically exhibit certain degrees of statistical self-similarity, with an example being the occurrence of power-law domain growth and dynamical scaling of the correlation functions [1]. These scaling behaviors are believed to be quite generic, and can be roughly categorized into a few universality classes. The dominant factors affecting the classification, among others, include spatial dimensionality, conservation properties, and the order-parameter symmetry. Most studies in this area have focused on systems involving spontaneous symmetry breaking. In such systems, it has been recognized that topological defects, i.e., singularities in the local order parameter, play an important role in controlling the long-time dynamics. In fact, all major features of the growth kinetics can be interpreted in terms of the behavior of these topological defects [2,3].

In this paper, we investigate the ordering dynamics in the kinetic Ising gauge models, which, even though exhibiting genuine phase transitions, do not have a local order parameter. These systems are unusual since they are invariant under a *local* gauge symmetry which cannot be spontaneously broken [4]. This work is partly motivated by the recent observation [5] that systems involving gauge fields show interesting features in dynamics. We shall illustrate that, although the absence of a local order parameter precludes the characterization of topological defects using the usual homotopy analysis, it is still helpful to find the low-energy topological excitations that are the disordering agents of the system. In the Ising gauge models, they are identified as frustrated plaquettes or frustrated plaquette strings. These objects, being gauge invariant, dominate the ordering kinetics asymptotically. While dynamical scaling is still found to hold during phase transition, the coarsening of typical domain size follows a unique growth law $L(t)\sim(t/\ln t)^{1/2}$ in both two and three dimensions. We suggest, in particular, a duality between the asymptotic kinetics of the two-dimensional (2D) Ising gauge model and that of 2D annihilating random walks.

Consider an infinite d-dimensional hypercubic lattice with Ising spins $\sigma=\pm1$ defined on each *link*. The spins couple ferromagnetically to each other, and are all coupled to a heat bath at temperature T. Let the pair $(\mathbf{r},\alpha)$ label a link with $\mathbf{r}$ being the coordinate of a site and α denoting a unit lattice vector. Equilibrium properties of the system are uniquely determined by assuming the Hamiltonian to take the Ising form:

$$\mathcal{H}(\{\sigma\})=-J\sum_{\mathbf{r},\alpha,\beta}\sigma_{\mathbf{r},\alpha}\sigma_{\mathbf{r}+\alpha,\beta}\sigma_{\mathbf{r}+\beta,\alpha}\sigma_{\mathbf{r},\beta}\ , \tag{1}$$

where the four-spin products are formed around the plaquettes of the lattice and $J>0$.

This Ising gauge model differs from the usual Ising model in that it has a local symmetry while the latter has a global symmetry. Indeed, the Hamiltonian (1) is invariant under the gauge transformation $\sigma_{\mathbf{r},\alpha}\rightarrow g_{\mathbf{r}}\sigma_{\mathbf{r},\alpha}g_{\mathbf{r}+\alpha}$, with $g_{\mathbf{r}}=\pm1$, corresponding to the local operation of flipping all the spins connected to a lattice site. The above model belongs to a more general class of lattice gauge models first proposed by Wegner [6]. Its equilibrium properties are well known [4]. In two dimensions, exact duality exists between the partition function of the Ising gauge model and that of the uncoupled 1D Ising models. Thus the 2D Ising gauge model has a phase transition strictly at $T=0$. Another duality between the 3D Ising gauge model and the 3D Ising model proves the existence of a finite-temperature second-order phase transition. For $d\geq4$, numerical evidence suggests that the model has first-order transitions at $T\neq0$.

Dynamics are incorporated into the model by assigning a transition rate for single spin flip $\sigma_{\mathbf{r},\alpha}\rightarrow-\sigma_{\mathbf{r},\alpha}$. We use a rate of the Glauber form [7]:

$$W(\sigma_{\mathbf{r},\alpha}\rightarrow-\sigma_{\mathbf{r},\alpha})=\tfrac{1}{2}[1-\sigma_{\mathbf{r},\alpha}\tanh E(\{\sigma\}_{\mathbf{r},\alpha})]\ , \tag{2}$$

where $\{\sigma\}_{\mathbf{r},\alpha}$ denotes the set of all spins except $\sigma_{\mathbf{r},\alpha}$ and

$$E(\{\sigma\}_{\mathbf{r},\alpha})=(J/k_BT)\sum_{\beta}\sigma_{\mathbf{r}+\alpha,\beta}\sigma_{\mathbf{r}+\beta,\alpha}\sigma_{\mathbf{r},\beta}\ , \tag{3}$$

with the β summation restricted to unit vectors distinct from α. Rate (2) satisfies detailed balance.

We first study the coarsening dynamics of the Ising gauge model in two dimensions. Since the phase transition occurs strictly at zero temperature, we consider the dynamics of the system following a rapid quench from the disordered state at $T=\infty$ to $T=0$. To proceed, we must identify the proper disordering agents in the phase transition. It is easy to see that the gauge model has an infinitely degenerate ground state at $T=0$. In fact, any spin configuration that is a pure gauge $\sigma_{\mathbf{r},\alpha}=g_{\mathbf{r}}g_{\mathbf{r}+\alpha}$ is a ground state. In the ground state, any four-spin product around a plaquette has to be unity: $\sigma\sigma\sigma\sigma=1$. Apparently, the lowest-energy topological excitation of the ground state corresponds to the spin configuration with a "frustrated" plaquette, around which the product of spins equals $\sigma\sigma\sigma\sigma=-1$. The frustrated plaquettes, which are themselves gauge-invariant objects, play the roles of topological defects in governing the asympotoic ordering kinetics. The dynamics of one frustrated plaquette is easy to visualize once we realize that (2) takes a simpler form at $T=0$,

$$W(\sigma_{\mathbf{r},\alpha}\rightarrow-\sigma_{\mathbf{r},\alpha}) =\frac{1}{2}\left[1-\tfrac{1}{2}\sigma_{\mathbf{r},\alpha}\sum_{\beta}\sigma_{\mathbf{r}+\alpha,\beta}\sigma_{\mathbf{r}+\beta,\alpha}\sigma_{\mathbf{r},\beta}\right] . \quad (4)$$

Imagine one frustrated plaquette and its neighborhood. By flipping any one of the four spins around the plaquette according to (4), the plaquette can either move to a neighboring position or annihilate with another nearby frustrated plaquette. In addition, at $T=0$, no new frustrated plaquettes will be created during the phase transition. Thus the ordering dynamics is a continuous process of the motion and annihilation of the frustrated plaquettes remnant from the initial state. A characteristic length scale of the system, $L(t)$, can consequently be introduced as the average distance between frustrated plaquettes.

The preceeding discussion hints at a possible connection of the current problem to the problem of 2D annihilating random walks (ARW's) [8]. Indeed, we suggest that the asymptotic behaviors of the 2D kinetic Ising gauge model are identical to those of the latter problem, once we regard the plaquettes as diffusing, "point-like" particles. Our claim is supported by numerical simulations of the gauge model detailed below. It is worth pointing out that this relation between the two problems is an extension of the well-known duality between the zero-temperature 1D kinetic Ising model and the 1D annihilating random walks [9]. An n-dimensional ARW, also known in chemical kinetics as the diffusion-limited reaction [10], $A+A\rightarrow$inert, is defined as the process of identical particles performing simple random walks and annihilating upon encounter. The ARW, as one of the simplest interacting-particle systems [11], has been intensively studied. One quantity of interest, the asymptotic mean density $\rho(t)$ of remaining particles in n dimensions, has been rigorously shown [12] to behave as

$$\rho(t)\sim\begin{cases}\dfrac{\ln t}{t} & \text{as } t\rightarrow\infty\ ,\quad n=2\\ t^{-1} & \text{as } t\rightarrow\infty\ ,\quad n\geq 3\ .\end{cases} \quad (5)$$

The presence of a logarithmic term for $n=2$ reflects the relevance of spatial fluctuations in the walker density [10] and is intimately related to the weak recurrence property of 2D simple random walks [12].

Returning to the gauge model, we introduce a nonlocal, gauge-invariant operator measuring the density of frustrated plaquettes:

$$\mu_{\mathbf{x}}=\tfrac{1}{2}(1-\sigma\sigma\sigma\sigma)\ , \quad (6)$$

where the four-spin product is around a plaquette centered at position $\mathbf{x}$. The average density of frustrated plaquettes is then given by $\rho(t)=\langle\mu_{\mathbf{x}}\rangle$. We have carried out numerical simulations of the 2D gauge model on square lattices of size 1000×1000. In Fig. 1, we show the gauge-invariant density of frustrated plaquettes, obtained from averages over 10 sets of random initial conditions, as a function of time. To extract the asymptotic behavior of $\rho(t)$, it is beneficial to plot the quantity $t\rho/\log_{10}t$ vs $1/\log_{10}t$, as shown in the inset. The reason for plotting the data in such a way is to exclude the possibility of a pure power-law decay without the logarithmic term. Clearly $\rho(t)\sim1/t$ is excluded since this would lead to a curve approaching the origin at smaller $1/\log_{10}t$. Similarly asymptotic behaviors of the type $\rho(t)\sim1/t^{\alpha}$ with $\alpha\neq1$ are also inconsistent with the data shown in the inset. Therefore we conclude from Fig. 1 that $t\rho/\log_{10}t\rightarrow\text{const}>0$ as $t\rightarrow\infty$, which implies that

$$\rho(t)\sim\frac{\ln t}{t} \quad \text{as } t\rightarrow\infty\ , \quad (7)$$

asymptotically, just as in the 2D ARW. As a result, the characteristic length scale $L(t)\sim\rho^{-1/2}\sim(t/\ln t)^{1/2}$.

Further insights about the ordering process can be gained by studying the proper correlation functions. Note that, just like in equilibrium, average of all non-gauge-invariant quantities vanish. We concentrate on the plaquette-plaquette correlation function defined as $C(\mathbf{x}-\mathbf{x}',t)\equiv\langle\mu_{\mathbf{x}}\mu_{\mathbf{x}'}\rangle$. Figure 2 is the numerically calculated correlation function. It is found that the correlation function satisfies dynamical scaling of the form

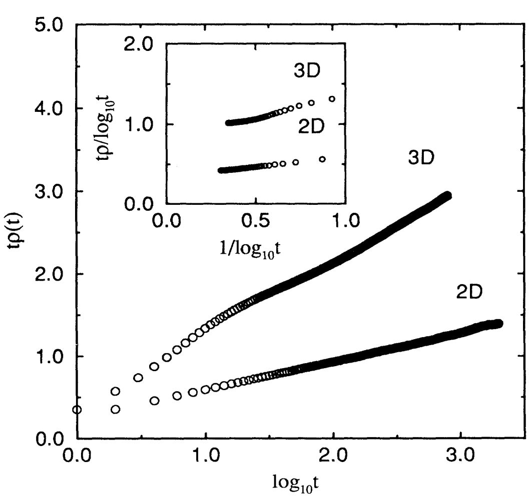

FIG. 1. Average density of frustrated plaquettes $\rho(t)$ in two- and three-dimensions. Curves in the inset are plotted in the way, $t\rho/\log_{10}t$ vs $1/\log_{10}t$, to illustrate the asympotic behavior $\rho\sim\log_{10}t/t$ as $t\rightarrow\infty$.

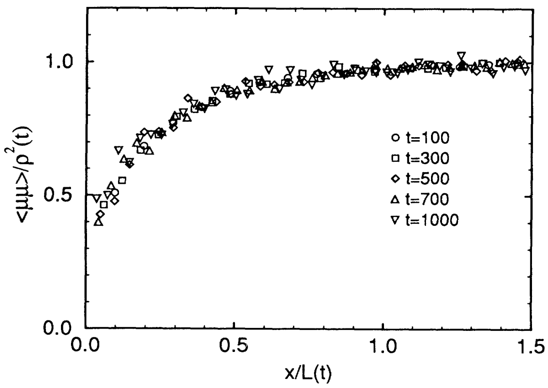

FIG. 2. Dynamical scaling of the plaquette-plaquette correlation function in two dimensions. Data are averaged over 64 initial conditions on 512×512 lattices.

$$C(\mathbf{x}-\mathbf{x}',t)=\rho^2(t)F\left[\frac{|\mathbf{x}-\mathbf{x}'|}{L(t)}\right] , \tag{8}$$

where F is a universal function. For comparison, we have also computed the two-particle correlation function of the ARW, which, if superimposed on Fig. 2, would be indistinguishable from that of the gauge model. These calculations lend further support to our claim that the 2D gauge model is in the same universality class as the ARW. No analytic determination of $F(x)$ is yet available. However, we may determine its short-distance behavior rather easily. Starting from the general master equation for the probability density of spins and using the rate (4), we derive the following equation for the plaquette moments:

$$\frac{\partial}{\partial t}\langle\mu_{\mathbf{x}}\rangle=\frac{1}{2}\sum_{\alpha}\langle\mu_{\mathbf{x}+\alpha}\rangle-2\langle\mu_{\mathbf{x}}\rangle-\sum_{\alpha}\langle\mu_{\mathbf{x}}\mu_{\mathbf{x}+\alpha}\rangle . \tag{9}$$

Equations for higher moments in general show a more complex hierarchy. After translational average using space homogeneity, we have

$$\frac{d}{dt}\rho(t)=-4C(1,t) . \tag{10}$$

Using (7), we obtain

$$C(1,t)\sim\frac{\ln t}{t^2} \quad \text{as } t\to\infty . \tag{11}$$

Assuming dynamical scaling of the form (8), Eq. (11) gives the short-distance behavior of the scaling function

$$F(x)\sim-\frac{1}{\ln x} \quad \text{for } x\to 0 . \tag{12}$$

At long distances, we have trivially $F(x)=1$ as $x\to\infty$.

Now we proceed to study the coarsening dynamics in three dimensions. It is still obvious that frustrated plaquettes disorder the ground state. However, there is an additional feature pertaining to the 3D system. Consider the six plaquettes enclosing a unit cube. The product of frustrations of these plaquettes is unity,

$$\prod_{i=1}^{6}(\sigma\sigma\sigma\sigma)_i=1 , \tag{13}$$

since each spin is multiplied twice. Therefore the number of frustrated plaquettes enclosing each unit cube has to be even. This constraint introduces a continuity property that the frustrated plaquettes have to connect to form stringlike structures. Here, for each frustrated plaquette, we define a unit string segment passing through the center of the plaquette. These "strings," either forming a loop or extending to infinity, are the topological excitations determining the asymptotic behavior of the 3D ordering kinetics.

Let $\rho(t)$ denote the total length of the string network and define the characteristic length of the system $L(t)$ as the average interstring spacing such that $L(t)\equiv 1/\sqrt{\rho(t)}$. In Fig. 1(b), we show the behavior of $\rho(t)$ obtained from numerical simulations on a 64^3 lattice, averaged over 60 initial conditions. Again, we find that

$$\rho(t)\sim\frac{\ln t}{t} , \tag{14}$$

which implies that the typical length scale $L(t)\sim(t/\ln t)^{1/2}$. The presence of a logarithmic term in 3D coarsening is somewhat unexpected. It suggests that the ordering dynamics in this case is not driven purely by the curvature of the plaquette strings, but that spatial fluctuations in the string density are also relevant. We have not studied higher-dimensional orderings, but it is reasonable to believe that this result will remain valid for all $d\geq 2$.

The model we considered belongs to a more general category of kinetic Ising gauge models, M_{dn} ($d\geq n$), using Wegner's notation [6]. In model M_{dn}, the Ising spins are defined on all the unit $(n-1)$-cubes of an infinite d-dimensional hypercubic lattice, and the Hamiltonian (1) is replaced by a summation of the $2n$-spin products around unit n-cubes. Dynamics are again incorporated using Glauber's rates. The model we considered in this paper corresponds to $n=2$. Now, extending our previous arguments to these more general models, we reach the general conclusion that the asymptotic kinetics of the zero-temperature M_{nn} model is dual to that of the n-dimensional annihilating random walks for all $n\geq 1$. Consequently, the density of frustrated n-cubes is, according to (5), $\rho(t)\sim t^{-1}$ for $n\geq 3$. Defining the characteristic length scale of the system accordingly, as $L(t)\sim\rho^{-1/n}$, we conclude that the system coarsens as $L(t)\sim t^{1/n}$. This is true for all $d\geq n\geq 3$.

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